Classification of affine operators up to biregular conjugacy

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Abstract

Let $f(x) = Ax + b$ and $g(x) = Cx + d$ be two affine operators given by n-by-n matrices A and C and vectors b and d over a field $\mathbb{F}$. They are said to be biregularly conjugate if $f = h^{-1}gh$ for some bijection $h : \mathbb{F}^n \to \mathbb{F}^n$ being biregular, this means that the coordinate functions of h and h^{-1} are polynomials. Over an algebraically closed field of characteristic 0, we obtain necessary and sufficient conditions of biregular conjugacy of affine operators and give a canonical form of an affine operator up to biregular conjugacy. These results for bijective affine operators were obtained by J. Blanc [Conjugacy classes of affine automorphisms of $\mathbb{K}^n$ and linear automorphisms of $\mathbb{P}^n$ in the Cremona groups, Manuscripta Math. 119 (2006) 225–241].

AMS classification: 37C15; 15A21

Keywords: Affine operators; Biregular conjugacy; Canonical forms

1 Introduction and theorem

In this article, affine operators are classified up to biregular conjugacy. All matrices and vector spaces that we consider are over an algebraically closed field $\mathbb{F}$ of zero characteristic.

A map $f : \mathbb{F}^n \to \mathbb{F}^n$ of the form $f(x) = Ax + b$ with $A \in \mathbb{F}^{n\times n}$ and $b \in \mathbb{F}^n$ is called an *affine operator* and A is called its *matrix*. Two maps $f,\ g : \mathbb{F}^n \to \mathbb{F}^n$ are *biregularly conjugate* if $g = h^{-1}fh$ for some bijection $h : \mathbb{F}^n \to \mathbb{F}^n$ being *biregular*, this means that h and h^{-1} have the form

$$(x_1, \dots, x_n) \mapsto (h_1(x_1, \dots, x_n), \dots, h_n(x_1, \dots, x_n)),$$

in which all h_i are polynomials over $\mathbb{F}$. The group of biregular operators on $\mathbb{F}^n$ is called the *affine Cremona group*.

Two matrices $A, B \in \mathbb{F}^{n\times n}$ are *similar* if $A = S^{-1}BS$ for some nonsingular $S \in \mathbb{F}^{n\times n}$. An element x is called a *fixed point* of f if $f(x) = x$. An affine operator without fixed point can not be biregularly conjugate to an affine operator with fixed point since if two maps on $\mathbb{F}^n$ are biregularly conjugate then they have the same number of fixed points.

Jérémy Blanc classified bijective affine operators up to biregular conjugacy as follows.

Theorem 1 (Blanc [1]). *Let $\mathbb{F}$ be an algebraically closed field of characteristic* 0*.*

(a) *Two bijective affine operators $\mathbb{F}^n \to \mathbb{F}^n$ with fixed points are biregularly conjugate if and only if their matrices are similar.*

(b) *Each bijective affine operator $f : \mathbb{F}^n \to \mathbb{F}^n$ without fixed point is biregularly conjugate to an "almost-diagonal" affine operator*

$$(x_1, x_2, \dots, x_n) \mapsto (x_1 + 1, \alpha_2 x_2, \dots, \alpha_n x_n), \tag{1}$$

in which $1, \alpha_2, \dots, \alpha_n \in \mathbb{F} \setminus 0$ are all eigenvalues of the matrix of f repeated according to their multiplicities. The affine operator (1) *is uniquely determined by f, up to permutation of $\alpha_2, \dots, \alpha_n$.*

Each square matrix

$$A \quad \text{is similar to} \quad A_* \oplus A_\circ, \tag{2}$$

in which A_* is nonsingular and $A_\circ$ is nilpotent.

In this article, we proof the following theorem, in which Theorem 1 is extended to arbitrary affine operators.

Theorem 2. *Let $\mathbb{F}$ be an algebraically closed field of characteristic* 0*.*

(a) *The criterion of biregular conjugacy of two affine operators* $f(x) = Ax + b$ *and* $g(x) = Cx + d$ *over* $\mathbb{F}$ *is the following:*

   (i) *If* f *and* g *have fixed points, then* f *and* g *are biregularly conjugate if and only if their matrices* A *and* C *are similar.*

   (ii) *If* f *and* g *have no fixed points, then* f *and* g *are biregularly conjugate if and only if the nonsingular matrices* A_* *and* C_* *(defined in (2)) have the same eigenvalues with the same multiplicities, and the nilpotent matrices* $A_\circ$ *and* $C_\circ$ *are similar.*

(b) *A canonical form of an affine operator* $f(x) = Ax + b$ *under biregular conjugacy is the following:*

   (i) *If* f *has a fixed point, then* f *is biregularly conjugate to the linear operator* $x \mapsto Jx$*, in which* J *is the Jordan canonical form of* A *uniquely determined by* f *up to permutation of blocks.*

   (ii) *If* f *has no fixed point, then* f *is biregularly conjugate to*

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \mapsto \begin{bmatrix} x_1 + 1 \\ \alpha_2 x_2 \\ \vdots \\ \alpha_k x_k \\ J_\circ \tilde{x} \end{bmatrix}, \qquad \tilde{x} := \begin{bmatrix} x_{k+1} \\ x_{k+2} \\ \vdots \\ x_n \end{bmatrix}, \tag{3}$$

   in which $1, \alpha_2, \ldots, \alpha_k \in \mathbb{F} \setminus 0$ *are all the eigenvalues of* A_* *repeated according to their multiplicities, and* $J_\circ$ *is the Jordan canonical form of* $A_\circ$*. The affine operator (3) is uniquely determined by* f*, up to permutation of eigenvalues* $\alpha_2, \ldots, \alpha_k$ *and permutation of blocks in* $J_\circ$*.*

2 Proof of Theorem 2

Lemma 1 ([1, 2])**.** *An affine operator* $f(x) = Ax + b$ *over* $\mathbb{F}$ *has a fixed point if and only if* f *is biregularly conjugate to its linear part* $f_{\text{lin}}(x) := Ax$.

Proof. If p is a fixed point of f, then f and f_{lin} are biregularly conjugate via $h(x) := x + p$ since for each x

$$\begin{aligned}(h^{-1} f h)(x) &= (h^{-1} f)(x + p) = h^{-1}(A(x + p) + b) \\ &= h^{-1}(Ax + (p - b) + b) = h^{-1}(Ax + p) = Ax = f_{\text{lin}}(x)\end{aligned}$$

and so $f_{\mathrm{lin}} = h^{-1} f h$.

Conversely, let f and f_{lin} be biregularly conjugate. Since $f_{\mathrm{lin}}(0) = 0$, f_{lin} has a fixed point. Because biregularly conjugate maps have the same number of fixed points, f has a fixed point too. $\square$

It is convenient to give an affine operator $f(x) = Ax + b$ by the pair (A, b) and identify f with this pair.

For two affine operators f on $\mathbb{F}^m$ and g on $\mathbb{F}^n$, their *direct sum* is the affine operator $f \oplus g$ on $\mathbb{F}^{m+n}$ defined as follows:

$$(f \oplus g)(\begin{bmatrix} x \\ y \end{bmatrix}) := \begin{bmatrix} f(x) \\ g(y) \end{bmatrix}.$$

Thus,

$$(A, b) \oplus (C, d) = \left(\begin{bmatrix} A & 0 \\ 0 & C \end{bmatrix}, \begin{bmatrix} b \\ d \end{bmatrix} \right).$$

We write $f \sim g$ if f and g are biregularly conjugate. Clearly,

$$f \sim f' \text{ and } g \sim g' \quad \Longrightarrow \quad f \oplus g \sim f' \oplus g'. \tag{4}$$

The statement (i) of Theorem 2(a) can be proved as [1, Proposition 2]: by Lemma 1, affine operators $f(x) = Ax + b$ and $g(x) = Cx + d$ with fixed points are biregularly conjugate if and only if their linear parts $f_{\mathrm{lin}}(x) = Ax$ and $g_{\mathrm{lin}}(x) = Cx$ are biregularly conjugate if and only if there exists a biregular map $\varphi : \mathbb{F}^n \to \mathbb{F}^n$ such that $\varphi f_{\mathrm{lin}} = g_{\mathrm{lin}} \varphi$. Differentiating the last equality and evaluating at zero gives $D\varphi(0)\, A = C\, D\varphi(0)$, thus A and C are similar. Conversely, if $A = S^{-1} C S$ then $\psi f_{\mathrm{lin}} = g_{\mathrm{lin}} \psi$ with $\psi(x) := Sx$.

The statement (i) of Theorem 2(b) follows from (i) of Theorem 2(a).

In the remaining part of the paper, we prove the statement (ii) of Theorem 2(b), which ensures (ii) of Theorem 2(a).

2.1 Reduction to the canonical form

Let $f(x) = Ax + b$ be an affine operator over $\mathbb{F}$ without fixed point. Let A_* and $A_\circ$ be defined by (2). The map f can be reduced to (3) by transformations of biregular conjugacy in 3 steps:

- f is reduced to the form

$$(A_*, c) \oplus (J_\circ, s), \tag{5}$$

in which $J_\circ$ is the Jordan canonical form of $A_\circ$. We make this reduction by a transformation of *linear conjugacy* $f \mapsto h^{-1}fh$, given by a linear map $h(x) = Sx$ with a nonsingular matrix S. Then $(A, b) \mapsto (S^{-1}AS, S^{-1}b)$; we take S such that $S^{-1}AS = A_* \oplus J_\circ$.

- (5) is reduced to the form

$$(A_*, c) \oplus (J_\circ, 0). \tag{6}$$

  We use (4) and $(J_\circ, s) \sim (J_\circ, 0)$, the latter holds by Lemma 1. Note that (A_*, c) has no fixed point since if $(x_1, \dots, x_k)$ is a fixed point of (A_*, c) then $(x_1, \dots, x_k, 0, \dots, 0)$ is a fixed point of (6), but f has no fixed point.

- (6) is reduced to the form

$$(D_\alpha, e_1) \oplus (J_\circ, 0), \tag{7}$$

  in which

$$D_\alpha := \begin{bmatrix} 1 & & & 0 \\ & \alpha_2 & & \\ & & \ddots & \\ 0 & & & \alpha_k \end{bmatrix}, \quad e_1 := \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

  and $1, \alpha_2, \dots, \alpha_k$ are all the eigenvalues of A_* which are repeated according to their multiplicities (note that (7) is another form of (3)). We use (4) and $(A_*, c) \sim (D_\alpha, e_1)$, the latter follows from Theorem 1(b).

2.2 Uniqueness of the canonical form

Let

$$f = f_* \oplus f_\circ, \quad f_* = (D_\alpha, e_1) : \mathbb{F}^p \to \mathbb{F}^p, \quad f_\circ = (J_\circ, 0) : \mathbb{F}^{n-p} \to \mathbb{F}^{n-p},$$

and

$$g = g_* \oplus g_\circ, \quad g_* = (D_\beta, e_1) : \mathbb{F}^q \to \mathbb{F}^q, \quad g_\circ = (J'_\circ, 0) : \mathbb{F}^{n-q} \to \mathbb{F}^{n-q},$$

be two affine operators of the form (7) on $\mathbb{F}^n$, in which f_* and g_* have no fixed points. Let f and g be biregularly conjugate. For each $i = 1, 2, \dots,$ the images of f^i and g^i are the sets

$$V_i := f^i\mathbb{F}^n = \mathbb{F}^p \oplus J_\circ^i\mathbb{F}^{n-p}, \qquad W_i := g^i\mathbb{F}^n = \mathbb{F}^q \oplus J_\circ'^i\mathbb{F}^{n-q},$$

and so they are vector subspaces of $\mathbb{F}^n$ of dimensions

$$\dim V_i = p + \operatorname{rank} J_\circ^i, \qquad \dim W_i = q + \operatorname{rank} J_\circ'^i. \tag{8}$$

Since f and g are biregularly conjugate, there exists a biregular map $h : \mathbb{F}^n \to \mathbb{F}^n$ such that $f = h^{-1}gh$. Then

$$hf^i = g^i h, \quad hf^i\mathbb{F}^n = g^i h\, \mathbb{F}^n = g^i\mathbb{F}^n, \quad h\, V_i = W_i. \tag{9}$$

By [3, Chapter 1, Corollary 3.7], the last equality implies

$$\dim V_i = \dim W_i, \qquad i = 1, 2, \ldots$$

If $m := \max(n-p, n-q)$, then $J_\circ^m = J_\circ'^m = 0$, and so

$$p = \dim V_m = \dim W_m = q.$$

Thus, f_* and g_* are affine bijections $V_* \to V_*$ on the same space

$$V_* := V_m = W_m = \mathbb{F}^p.$$

By (9), the restriction of h to V_* gives some biregular map $h_* : V_* \to V_*$. Restricting the equality $hf = gh$ to V_*, we obtain $h_*f_* = g_*h_*$. Therefore, f_* and g_* are biregularly conjugate. By Theorem 1, their matrices D_α and D_β *coincide up to permutation of eigenvalues*.

The nilpotent Jordan matrices $J_\circ$ and $J_\circ'$ *coincide up to permutation of blocks* since by (8) the number of their Jordan blocks is equal to $n - \dim V_1$, the number of their Jordan blocks of size ≥ 2 is equal to $(n - \dim V_2) - (n - \dim V_1)$, the number of their Jordan blocks of size ≥ 3 is equal to $(n - \dim V_3) - (n - \dim V_2)$, and so on.

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